Guess your neighbour's input:
a multipartite non-local game with no quantum advantage

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We present a multipartite nonlocal game in which each player must guess the input received by his neighbour. We show that quantum correlations do not perform better than classical ones at this game, for any prior distribution of the inputs. There exist, however, input distributions for which general no-signalling correlations can outperform classical and quantum correlations. Some of the Bell inequalities associated to our construction correspond to facets of the local polytope. Thus our multipartite game identifies parts of the boundary between quantum and post-quantum correlations of maximal dimension. These results suggest that quantum correlations might obey a generalization of the usual no-signalling conditions in a multipartite setting.

In recent years, the study and understanding of quantum nonlocality – the fact that certain quantum correlations violate Bell inequalities [1] – has benefited from a cross-fertilization with information concepts.

On one hand, nonlocality has been identified as a key resource for quantum information processing. It allows, for instance, the reduction of communication complexity [2], and in the device-independent scenario, where one wants to achieve an information task without any assumption on the devices used in the protocol, it can be exploited for secure key distribution [3], state tomography [4], and randomness generation [5].

On the other hand, information concepts have provided a deeper understanding of the nature of quantum nonlocality. It is known in particular that the no-signalling principle (no arbitrarily fast communication between remote parties) is compatible with the existence of correlations more nonlocal than those allowed in quantum theory [6, 7]. However, recent works have shown that the existence of such stronger-than-quantum correlations would have deep information-theoretic consequences: they would, for instance, collapse communication complexity [8] and allow perfect nonlocal computation [9]. In a related direction, it has been realized that quantum correlations actually obey a strengthened version of no-signalling, the principle of information causality [10].

Up to now, such questions have been almost exclusively considered in the bipartite scenario. Here our aim is to investigate the separation between quantum and no-signalling correlations in a multipartite scenario. For this, we introduce and study a simple multipartite nonlocal game, *Guess Your Neighbour's Input* (GYNI).

In GYNI, N distant players are arranged on a ring and each receive an input bit $x_i \in \{0, 1\}$ (see Fig. 1). The goal is that each participant provides an output bit
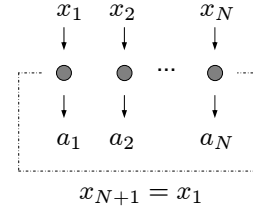

FIG. 1: Representation of the GYNI nonlocal game. The goal is that each party outputs its right-neighbour's input: $a_i = x_{i+1}$.

$a_i \in \{0, 1\}$ equal to its right neighbour's input bit:

$$a_i = x_{i+1} \qquad \text{for all } i = 1, \ldots, N, \qquad (1)$$

where $x_{N+1} \equiv x_1$. The 2^N possible input strings $\mathbf{x} = (x_1, \ldots, x_N)$ are chosen according to some prior distribution $q(\mathbf{x}) = q(x_1, \ldots, x_N)$, which is known to the parties. The figure of merit of the game is given by the average winning probability

$$\omega = \sum_{\mathbf{x}} q(\mathbf{x}) P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}), \qquad (2)$$

where $P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}) = P(a_1 = x_2, \ldots, a_N = x_1|x_1, \ldots, x_N)$ denotes the probability of obtaining the correct outputs (1) when the players have received the input string $\mathbf{x}$. Of course, players are not allowed to communicate after the inputs are distributed. Thus, their performance depends only on the initially agreed strategy and on the shared physical resources.

The GYNI game captures a particular notion of signalling: if the players were able to win with high probability, their output would reveal some information about their neighbour's input. We therefore expect that the nonlocal correlations of quantum theory cannot be exploited by non-communicating observers to perform better at GYNI than using classical resources alone. We confirm this intuition and prove that, indeed, quantum correlations provide no advantage over classical correlations.

Surprisingly, however, the no-signalling principle is not at the origin of the quantum limitation: for $N \geq 3$, there exist input distributions q for which no-signalling correlations provide an advantage over the best classical and quantum strategies. This suggests the possibility that in a multipartite scenario, quantum correlations obey a qualitatively stronger version of the usual no-signalling conditions.

Each of the input distributions q associated with a non-trivial no-signalling strategy defines a Bell inequality whose maximal classical and quantum values coincide, but whose no-signaling value is strictly larger. Interestingly, some of these inequalities define facets of the polytope of local correlations. We thus prove the existence of non-trivial facet Bell inequalities with no quantum violation, answering a question raised by Gill [11]. Moreover, since these Bell inequalities are facets, the GYNI game identifies a portion of the boundary of the set of quantum correlations of non-zero measure, in contrast with previous information-theoretic or physical limitations on nonlocality [8–10, 12–14].

GYNI with classical and quantum resources. We start by showing that the optimal classical and quantum winning strategies are identical for any prior distribution q of the inputs. Let us first show that there is a simple classical strategy achieving a winning probability

$$\omega_c = \max_{\mathbf{x}} [q(\mathbf{x}) + q(\bar{\mathbf{x}})] , \qquad (3)$$

where $\bar{\mathbf{x}}$ denotes the "negation" of the input string $\mathbf{x}$, $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_N)$ with $\bar{x}_i = x_i \oplus 1$, and $\oplus$ denotes addition modulo 2. This strategy is based on the following simple observation.

Let $\mathbf{y}$ be an arbitrary string. If $\mathbf{x} \neq \mathbf{y}, \bar{\mathbf{y}}$, there exists an i s.t. $x_i = y_i$ and $x_{i+1} \neq y_{i+1}$. $\qquad (4)$

Indeed, if this was not the case, we would have that for any i, either $x_i \neq y_i$ or $x_{i+1} = y_{i+1}$. But this would in turn imply that either $\mathbf{x} = \mathbf{y}$ or $\mathbf{x} = \bar{\mathbf{y}}$, in contradiction with the hypothesis.

Consider now a classical strategy specified by the string $\mathbf{y}$, where each party outputs the bit $a_i = y_{i+1}$ if it received the input y_i, and outputs $a_i = \bar{y}_{i+1}$ if it received $\bar{y}_i$. It obviously follows that $P(\mathbf{a}_i = \mathbf{y}_{i+1}|\mathbf{y}) = 1$ and $P(\mathbf{a}_i = \bar{\mathbf{y}}_{i+1}|\bar{\mathbf{y}}) = 1$. On the other hand, $P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}) = 0$ for all $\mathbf{x} \neq \mathbf{y}, \bar{\mathbf{y}}$. Indeed, from observation (4), there exists an i such that $x_i = y_i$, but for which $a_i = y_{i+1} \neq x_{i+1}$. The winning probability of this classical strategy is thus equal to $\omega = q(\mathbf{y}) + q(\bar{\mathbf{y}})$, which yields (3) if we take $\mathbf{y}$ to be $q(\mathbf{y}) + q(\bar{\mathbf{y}}) = \max_{\mathbf{x}}[q(\mathbf{x}) + q(\bar{\mathbf{x}})]$.

We now prove that there is no better quantum (hence classical) strategy. In the most general quantum protocol, the parties share an entangled state $|\psi\rangle$ and perform projective measurements on their subsystem dependent on their inputs x_i. They then output their measurement results a_i. Denoting $M_{a_i}^{x_i}$ the projection operator associated to the output a_i for the input x_i, the probability

that the N players produce the correct output is thus given by

$$P(a_1 = x_2, \ldots, a_N = x_1 | x_1, \ldots, x_N) = \langle M_{x_2}^{x_1} \otimes \ldots \otimes M_{x_1}^{x_n} \rangle ,$$

and the average winning probability is

$$\omega = \sum_{\mathbf{x}} q(\mathbf{x}) \langle M_{\mathbf{x}} \rangle , \qquad (5)$$

where we have written $M_{\mathbf{x}} = M_{x_2}^{x_1} \otimes \ldots \otimes M_{x_1}^{x_n}$ for short. The operators $M_{\mathbf{x}}$ satisfy the following properties

$$M_{\mathbf{x}}^2 = M_{\mathbf{x}} , \qquad (6)$$

and

$$M_{\mathbf{x}} M_{\mathbf{y}} = 0 \quad \text{if } \mathbf{x} \neq \mathbf{y}, \bar{\mathbf{y}} . \qquad (7)$$

The first property follows from the fact that the $M_{\mathbf{x}}$ are projection operators. The second property follows from the orthogonality relations $M_{a_i}^{x_i} M_{\bar{a}_i}^{x_i} = 0$ and observation (4). Note that protocols involving mixed states or general measurements can all be represented in the above form by expanding the dimensionality of the initial state.

We now show, using (6) and (7), that $\omega = \sum_{\mathbf{x}} q(\mathbf{x}) M_{\mathbf{x}} \leq \omega_c$, where $\leq$ should be understood as an operator inequality, i.e., $A \leq B$ means that $\langle A \rangle \leq \langle B \rangle$ for all $|\psi\rangle$. First note that $\sum_{\mathbf{x}} q(\mathbf{x}) M_{\mathbf{x}} \leq \sum_{\mathbf{x}} q'(\mathbf{x}) M_{\mathbf{x}}$, where $q'(\mathbf{x}) = q(\mathbf{x}) + (\omega_c - q(\mathbf{x}) - q(\bar{\mathbf{x}}))/2$ since by definition $\omega_c - q(\mathbf{x}) - q(\bar{\mathbf{x}}) \geq 0$. It is thus sufficient to consider weights q such that $q(\mathbf{x}) + q(\bar{\mathbf{x}}) = \omega_c$ for all $\mathbf{x}$. We can then write

$$\omega_c - \sum_{\mathbf{x}} q(\mathbf{x}) M_{\mathbf{x}} = \left[\sqrt{\omega_c} - \sum_{\mathbf{x}} \alpha_{\mathbf{x}} M_{\mathbf{x}} \right]^2$$
$$+ \frac{1}{2} \sum_{\mathbf{x}} [\beta_{\mathbf{x}} M_{\mathbf{x}} - \beta_{\bar{\mathbf{x}}} M_{\bar{\mathbf{x}}}]^2 \quad (8)$$

where $\alpha_{\mathbf{x}} = \sqrt{\omega_c} - q(\bar{\mathbf{x}})/\sqrt{\omega_c}$ and $\beta_{\mathbf{x}} = \sqrt{q(\mathbf{x})q(\bar{\mathbf{x}})/\omega_c}$. To verify this identity we only need to use (6), (7), and the fact that $q(\mathbf{x}) + q(\bar{\mathbf{x}}) = \omega_c$. Note now that the right hand-side of (8) is ≥ 0, since it is a sum of square involving only hermitian operators. This shows that $\sum_{\mathbf{x}} q(\mathbf{x}) M_{\mathbf{x}} \leq \omega_c$, as announced.

The inequality $\sum_{\mathbf{x}} q(\mathbf{x}) P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}) \leq \omega_c$ can be interpreted as a Bell inequality whose local and quantum bound coincide. It is well known that in order to achieve a Bell violation in quantum theory one must perform measurements corresponding to non-commuting operators. The above proof, however, does not distinguish non-commuting operators from ordinary, commuting numbers: it is based on the algebraic identity (8) which follows only from Eqs. (6) and (7), regardless of whether the $M_{\mathbf{x}}$'s commute or not. This explains why the classical and quantum bounds are identical.

GYNI with no-signalling resources. At first sight, it may seem that the quantum limitation on the GYNI

game arises from the no-signalling principle: if the players were able to win with high probability, their output would somehow depend on their neighbour's input. This motivates us to look at how players constrained only by the no-signalling principle perform at GYNI.

Formally, the no-signalling principle states that the marginal distribution $P(a_{i_1}, \ldots, a_{i_k}|x_{i_1}, \ldots, x_{i_k})$ for any subset $\{i_1, \ldots, i_k\}$ of the n parties should be independent of the measurement settings of the remaining parties [7], i.e., that

$$P(a_{i_1}, \ldots, a_{i_k}|x_1, \ldots, x_N) = P(a_{i_1}, \ldots, a_{i_k}|x_{i_1}, \ldots, x_{i_k})$$

This guarantees that any subset of the parties is unable to signal to the other by their choice of inputs.

We show in Appendix A that players constrained only by no-signalling have a bounded winning probability $\omega_{ns} \leq 2\omega_c$. They thus cannot win in general with unit probability at GYNI. Furthermore, for certain input distributions, such as the one where all input strings are chosen with equal weight $q(\mathbf{x}) = 1/2^N$, we show as expected that $\omega_{ns} = \omega_c$. That is, for uniform and completely uncorrelated inputs, any resource performing better than a classical strategy is necessarily signalling.

Surprisingly, this property is not general. There exist distributions $q(\mathbf{x})$ for which no-signalling strategies outperform classical and quantum strategies. Consider for instance the following input distribution

$$q(\mathbf{x}) = \begin{cases} 1/2^{N-1} & \text{if } x_1 \oplus \cdots \oplus x_{\hat{N}} = 0 \\ 0 & \text{otherwise}, \end{cases} \quad (9)$$

where $\hat{N} = N$ if N is odd and $\hat{N} = N - 1$ if N is even. It easily follows from the previous analysis that for classical and quantum resources, $\omega_c = 1/2^{N-1}$. We now prove, however, that no-signalling resources can achieve $\omega_{ns} = 4/3\,\omega_c$. Note that the distribution (9) can be interpreted as a promise that the sum of the inputs (modulo 2) is equal to zero. This prior knowledge does not yield any information to the parties about the value of their neighbour's input, yet it can be exploited by no-signalling correlations to outperform classical strategies.

We start by considering the case $N = 3$, for which

$$\begin{aligned} \omega = \frac{1}{4} &[P(000|000) + P(110|011) \\ &+ P(011|101) + P(101|110)], \end{aligned} \quad (10)$$

where $P(000|000) = P(a_1 = 0, a_2 = 0, a_3 = 0|x_1 = 0, x_2 = 0, x_3 = 0)$, and so on. Consider the first three terms in (10). The no-signaling principle implies that

$$P(000|000) \leq \sum_{a_3} P(00a_3|000) = \sum_{a_3} p(00a_3|001),$$

$$P(110|011) \leq \sum_{a_2} P(1a_20|011) = \sum_{a_2} p(1a_20|001), \quad (11)$$

$$P(011|101) \leq \sum_{a_1} P(a_111|101) = \sum_{a_1} p(a_111|001).$$

By normalization of probabilities, the sum of the right-hand sides of Eqs. (11) is upper-bounded by one, and thus $P(000|000) + P(110|011) + P(011|101) \leq 1$. Similar conditions are obtained for any of the four possible combination of three terms in Eq. (10). Summing over these possibilities, we find $3[P(000|000) + P(110|011) + P(011|101) + P(101|110)] \leq 4$, or in other words $\omega_{ns} \leq 4/3 \times 1/4 = 4/3\,\omega_c$. Furthermore the inequality is saturated only if the four probabilities appearing in (10) are all equal to $1/3$. It turns out that the remaining entries of the probability table $P(\mathbf{a}|\mathbf{x}) = P(a_1 a_2 a_3|x_1 x_2 x_3)$ can be completed in a way that is compatible with the no-signalling principle, i.e, the bound $\omega_{ns} \leq 4/3\,\omega_c$ is achievable. Up to relabeling of inputs and outputs, there exist two inequivalent classes of extremal no-signalling correlations achieving this winning probability (see Appendix B). One of them takes the form $P(\mathbf{a}|\mathbf{x}) = 2/3\,g(\mathbf{a}, \mathbf{x}) + 1/3\,g'(\mathbf{a}, \mathbf{x})$ where g and g' are the following boolean functions

$$\begin{aligned} g(\mathbf{a}, \mathbf{x}) =& a_1 a_2 a_3 (1 \oplus x_1)(1 \oplus x_2)(1 \oplus x_3) \\ g'(\mathbf{a}, \mathbf{x}) =& (1 \oplus a_1)(1 \oplus a_2)(1 \oplus a_3) \quad (12) \\ & \oplus x_1 a_2 a_3 \oplus a_1 x_2 a_3 \oplus a_1 a_2 x_3 \oplus x_1 x_2 x_3. \end{aligned}$$

From this definition, it is easy to verify that $P(a_1 a_2 a_3|x_1 x_2 x_3)$ satisfies the no-signalling conditions and achieves winning probability $\omega_{ns} = 1/3 = 4/3\,\omega_c$.

The existence of no-signaling correlations achieving $\omega_{ns} = 4/3\,\omega_c$ in the case $N = 3$ is enough to show that $\omega_{ns} \geq 4/3\,\omega_c$ for any $N \geq 3$. This can be seen as follows. Consider the situation in which the first three parties use the optimal strategy for $N = 3$ while the remaining parties simply output their input. In this case, all the terms in ω vanish, except the four terms $P(000, 0 \ldots 0|000, 0 \ldots 0)$, $P(110, 0 \ldots 0|011, 0 \ldots 0)$, $P(011, 1 \ldots 1|101, 1 \ldots 1)$, and $P(101, 1 \ldots 1|110, 1 \ldots 1)$, which are all equal to $1/3$.

Beyond these analytical results, we obtained the maximal no-signaling values of ω_{ns} up to $N = 7$ players using linear programming. The ratios ω_{ns}/ω_c of no-signalling to classical winning probabilities are $4/3$ for $N = 3, 4$, $16/11$ for $N = 5, 6$, and $64/42$ for $N = 7$, showing that for more parties there exist no-signaling correlations that can outperform the optimal no-signaling strategy for $N = 3$. (Note that it can be shown that the winning probability for an odd number N of parties is always equal to the winning probability for $N + 1$ players, see Appendix C).

GYNI Bell inequalities. The GYNI Bell inequalities $\sum_{\mathbf{x}} q(\mathbf{x})P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}_i) \leq \omega_c$ are not violated by quantum theory, but can be violated by more general no-signalling theories. In [11], Gill raised the question of whether there exist Bell inequalities which (i) feature this 'no quantum advantage' property and (ii) define facets of the polytope of local correlations. Here we give a positive answer to this question. We have checked that the GYNI inequalities defined by the distribution

(9) are facet-defining for $N \leq 7$ players. More generally, we verified that the inequalities defined by the distribution $q(\mathbf{x})$ having uniform support on $\bigoplus_{i=1}^{\hat{N}} x_i = 0$ are facet-defining for all $N \leq 7$. We conjecture that they are facet-defining for any number of parties. Note also that the polytope of local correlations for the case $N = 3$ (with binary inputs and outputs) was completely characterized in [16]; the inequality corresponding to (10) belongs to the class 10 of [16]. Geometrically, our result shows that the polytope of local correlations and the set of quantum correlations have in common faces of maximal dimension (we recall that a facet corresponds to a $(d-1)$-dimensional face of a d-dimensional polytope).

This also implies that GYNI is an information-theoretic game that identifies a portion of the boundary of quantum correlations which is of non-zero measure. To the best of our knowledge, all previously introduced information-theoretic or physical principles recovering part of the quantum boundary – including nonlocal computation [9], nonlocality swapping [12], information causality [10, 13], and macroscopic locality [14] – only single out a portion of zero-measure [15].

Discussion and open questions. Our work raises plenty of new questions. First, it would be interesting to understand the structure of those input distributions q leading to a gap between no-signaling and classical/quantum correlations (See Appendix A, for a class of distributions for which there is no gap). For instance, in the case of four parties, the distribution q having uniform support on $x_1 \oplus x_2 \oplus x_3 \oplus x_1 x_2 x_3 = 0$ leads to $\omega_{ns} = 4/3\,\omega_c$. However, the corresponding Bell inequality is not a facet. Another question is thus to single out, among all relevant input distributions, those corresponding to facet Bell inequalities. For three parties, it follows from [16] that the distribution (9) is the unique possibility.

A further interesting problem is whether there exist facet Bell inequalities with no quantum advantage in the bipartite case. Note that our GYNI inequalities are non-trivial only for $N \geq 3$; for the case $N = 2$, the classical and no-signalling bounds are always equal. In ref. [9], examples of bipartite Bell inequalities with no quantum advantage have been presented in the context of nonlocal computation. However, as mentioned earlier, none of the Bell inequalities associated to nonlocal computation has been proven to be facet-defining. We studied this question here and could prove that none of the simplest inequalities from [9] (corresponding to the family of inequalities specified by the parameters $n = 2, 3$ in [9]) are facet inequalities. The proof uses a mapping from these inequalities to the space of correlation inequalities for n parties, two settings and two outcomes, which was fully characterized in Ref. [17]; see Appendix D for a detailed proof. We conjecture that none of the Bell inequalities introduced in [9] are facet-defining.

Coming back to our original motivation, it would be interesting to get a deeper understanding of the structure and information-theoretic properties of the no-signaling correlations giving an advantage over classical/quantum correlations, for instance those associated to inequality (10). In particular, it would be interesting to understand if they can be exploited for other information tasks. Finally, our results suggest that the quantum limitation on the GYNI game might originate from a generalization of the no-signalling principle in a multipartite setting. Can this intuition be made concrete? Are there more general information tasks with no quantum advantage?

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APPENDIX A

Here we derive the upper bound $\omega_{ns} \leq 2\omega_c$ for the winning probability ω_{ns} of no-signalling strategies. We then show that $\omega_{ns} = \omega_c$ for all input distributions $q(\mathbf{x})$ such that $q(\mathbf{x}) \leq q(\mathbf{y}) = q(\bar{\mathbf{y}})$ for some input string $\mathbf{y}$. Such distributions include in particular the uniform distribution where all input strings are chosen with equal weight $q(\mathbf{x}) = 1/2^N$.

To start we derive the upper bound $\omega_{ns} \leq 2\omega_c$, valid for any distribution $q(\mathbf{x})$. From the definition (3), we have that $q(\mathbf{x}) \leq \omega_{\mathbf{c}}$ for every input string $\mathbf{x}$. This trivially leads to the upper-bound

$$\omega_{ns} \leq \omega_c \sum_{\mathbf{x}} P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}). \qquad (13)$$

Notice that this bound is only meaningful when the right-hand side is smaller than 1, since obviously $\omega_{ns} \leq 1$. We now show that for all no-signalling distributions $\sum_{\mathbf{x}} P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}) \leq 2$, from which the bound $\omega_{ns} \leq 2\omega_c$ immediately follows.

First note that from the no-signaling condition,

$$P(a_1, \ldots, a_{k-1}|x_1, \ldots, x_{k-1}) \leq P(a_1, \ldots, a_k|x_1, \ldots, x_k). \qquad (14)$$

We now write

$$\sum_{\mathbf{x}} P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x})$$
$$= \sum_{x_1, \ldots, x_N} P(a_1 = x_2, \ldots, a_N = x_1|x_1, \ldots, x_N)$$
$$\leq \sum_{x_1, \ldots, x_N} P(a_1 = x_2, \ldots, a_{N-1} = x_N|x_1, \ldots, x_{N-1})$$
$$= \sum_{x_1, \ldots, x_{N-1}} P(a_1 = x_2, \ldots, a_{N-2} = x_{N-1}|x_1, \ldots, x_{N-2})$$

where the inequality follows from (14) and in the last equality we used the no-signaling condition after summing over x_N. Iteratively performing this last step, we finally obtain

$$\sum_{\mathbf{x}} P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}) \leq \sum_{x_1, x_2} P(a_1 = x_2|x_1) \leq 2. \qquad (15)$$

We now analyze the no-signalling winning probability for distributions satisfying $q(\mathbf{x}) \leq q(\mathbf{y}) = q(\bar{\mathbf{y}})$ for some input string $\mathbf{y}$. Note that for such weights $\omega_c = q(\mathbf{y}) + q(\bar{\mathbf{y}}) = 2q(\mathbf{y})$, as easily follows from (3). We thus have

$$\omega_{ns} = \sum_{\mathbf{x}} q(\mathbf{x})P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}) \leq \frac{\omega_c}{2} \sum_{\mathbf{x}} P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}).$$

But, as we have shown above, $\sum_{\mathbf{x}} P(\mathbf{a}_i = \mathbf{x}_{i+1}|\mathbf{x}) \leq 2$ for all no-signalling distributions , and thus $\omega_{ns} \leq \omega_c$. Since any classical strategy is also a no-signalling strategy, it actually holds that $\omega_{ns} = \omega_c$.

APPENDIX B

Here we describe two inequivalent no-signaling correlations which attain $\omega_{ns} = 4/3\,\omega_c$ for the tripartite inequality (10). These correlations are extremal non-local boxes in the sense of being vertices of the no-signaling polytope for three parties and binary inputs and outputs [7].

Writing (a, b, c) for (a_1, a_2, a_3) and (x, y, z) for (x_1, x_2, x_3), we can write the first box as

$$P_1(a, b, c|x, y, z) = \frac{1}{3}f(a, b, c, x, y, z) \qquad (16)$$

where $f(a, b, c, x, y, z)$ is the boolean function

$$f(a, b, c, x, y, z) = (1 \oplus b \oplus x \oplus y \oplus xy)(1 \oplus c \oplus z) \\ \oplus a(1 \oplus y \oplus cy \oplus b(c \oplus z)). \qquad (17)$$

Similarly, we can write the second box as

$$P_2(a, b, c|x, y, z) = \frac{2}{3}g(a, b, c, x, y, z) + \frac{1}{3}g'(a, b, c, x, y, z) \qquad (18)$$

with g and g' the two boolean functions

$$g(a, b, c, x, y, z) = abc(1 \oplus x)(1 \oplus y)(1 \oplus z) \\ g'(a, b, c, x, y, z) = (1 \oplus a)(1 \oplus b)(1 \oplus c) \qquad (19) \\ \oplus xbc \oplus ayc \oplus abz \oplus xyz.$$

Among the boxes that are equivalent to P_1 under relabeling of parties, inputs, and outputs, a total of 24 of them violate maximally the Bell inequality (10), and similarly for 8 of those that are equivalent to P_2. Even though other tripartite no-signaling boxes (inequivalent to P_1 or P_2 under relabeling of parties, inputs, or outputs) violate the Bell inequality (10), those 32 boxes obtained from P_1 and P_2 are the unique ones that violate it maximally.

APPENDIX C

Here we show that for the input distribution (9), the no-signaling bound for an even number of parties $N + 1$ is always equal to the no-signaling bound for N parties. Start by considering $N+1$-GYNI game, where the first N players use the optimal strategy for the N-player case and player $N + 1$ outputs its input. They then achieve a no-signaling violation equal to the N case, which imposes the lower bound $\omega_{ns}(N + 1) \geq \omega_{ns}(N)$. But this is actually the best average success these $N + 1$ players can obtain. To see that, consider the game for $N+1$ parties. Allowing players N and $N+1$ to communicate can only increase the achievable value of $\omega_{ns}(N + 1)$. Indeed, in this situation the best strategy that player N can adopt is to output x_{N+1}, which was communicated to him by player $N + 1$, while player $N + 1$ needs to guess x_1 given x_N and x_{N+1}. Clearly, the knowledge of x_{N+1} is of no use for

him since this bit is completely uncorrelated with the rest of the input string. Consequently, the situation is analogous to having players $1, \ldots, N-1, N+1$ (i.e. all players except player N) play the game for N parties. Therefore $\omega_{ns}(N+1) \leq \omega_{ns}(N)$ and we have finally that $\omega_{ns}(N+1) = \omega_{ns}(N)$ for odd N.

APPENDIX D

In what follows, we derive a criterion that is necessarily satisfied by any facet-defining Bell inequality associated to the task of nonlocal computation (NLC) [9], and show that none of the NLC Bell inequalities for boolean functions of two and three input bits are facet-defining.

Nonlocal computation is a distributed task of two parties, where the goal is to compute a given boolean function $f(\mathbf{z})$ of an n-bit string $\mathbf{z}$. The input bit string is decomposed into two strings $\mathbf{x}$ and $\mathbf{y}$, such that $\mathbf{x} \oplus \mathbf{y} = \mathbf{z}$. The bit string $\mathbf{x}$ is sent to party A while the bit string $\mathbf{y}$ is sent to party B. Upon receiving their input bit strings, A and B each output a single bit, a and b respectively, such that the following relation holds: $a \oplus b = f(\mathbf{z})$. Importantly, each party has locally no information about the input bit string $\mathbf{z}$, that is $P(x_i = z_i) = 1/2$ for all $i = 1, ..., n$. For each n, $f(\mathbf{z})$, and distribution of inputs $\tilde{p}(\mathbf{z})$, we obtain a Bell expression whose value is associated to the probability of success at the task. These NLC inequalities have the form

$$I(n, f, \tilde{p}) = \sum_{\mathbf{z}} (-1)^{f(\mathbf{z})} \tilde{p}(\mathbf{z}) \sum_{\mathbf{x} \oplus \mathbf{y} = \mathbf{z}} \langle A_x B_y \rangle \leq k(n, f, \tilde{p}) \tag{20}$$

where A_x and B_y are observables which take values $\{-1, 1\}$. Notice that each party measures 2^n observables.

In Ref. [9] it is proven that the best classical strategy is given by $A_x = (-1)^{a_x}$ and $B_y = (-1)^{b_y}$ with

$$a_x = \mathbf{u} \cdot \mathbf{x}, \qquad b_y = \mathbf{u} \cdot \mathbf{y} \oplus \delta, \tag{21}$$

where δ denotes a single bit and $\mathbf{u}$ an n-bit string shared by the parties. This classical strategy, which is a linear approximation of the function f, achieves a winning probability as high as any quantum resource. Thus the local and quantum bounds of inequalities (20) coincide. There exist however no-signaling correlations which can perform with winning probability one at this game.

Checking whether the NLC inequalities (20) are facet-defining is in general a hard problem since one should consider any input size n, boolean function f, and distribution $\tilde{p}(\mathbf{z})$. Below we give a first simplification to this problem by deriving a necessary criterion satisfied by facet NLC inequalities. Our method is based on a mapping from the $(2, 2^n, 2)$ correlation space – i.e. (2 parties, 2^n settings, 2 outcomes) – in which the NLC inequalities are defined, into the $(n, 2, 2)$ full-correlation

space for which the complete set of tight Bell inequalities has been provided in Ref. [17].

To any inequality of the form (20) defined by the triple $(n, f, \tilde{p})$, we associate the following Bell inequality in the $(n, 2, 2)$ full-correlation space:

$$I_{n22}(n, f, \tilde{p}) = \sum_{\mathbf{z}} c(\mathbf{z}) \langle C_{z_1} \ldots C_{z_n} \rangle \leq 2^{-n} k(n, f, \tilde{p}) \tag{22}$$

where $c(\mathbf{z}) = (-1)^{f(\mathbf{z})} \tilde{p}(\mathbf{z})$, and where we view $z_i \in \{0, 1\}$ as denoting one of two possible observables C_{z_i} of party i taking values $\{-1, 1\}$ (with $i = 1, \ldots, n$).

Lemma. *If the NLC inequality $I(n, f, \tilde{p})$ for n bits is facet-defining, then the corresponding inequality $I_{n22}(n, f, \tilde{p})$ is facet-defining in the $(n, 2, 2)$ full-correlation space.*

Proof. The deterministic extremal points of the $(n, 2, 2)$ full-correlation polytope are of the form [17]

$$\langle C_{z_1} \ldots C_{z_n} \rangle = (-1)^{u_1 z_1} \ldots (-1)^{u_n z_n} (-1)^{\delta} = (-1)^{\mathbf{u} \cdot \mathbf{z} \oplus \delta} \tag{23}$$

where $u_i \in \{0, 1\}$ specifies the local strategy of each party and $\delta \in \{0, 1\}$ represents an additional global sign flip, which we can think of as being carried out by the last party. These deterministic points are thus specified by the single bit δ and the n-bit string $\mathbf{u}$, and are therefore in one-to-one correspondence with the extremal points (21) saturating the inequalities (20). For any such strategy specified by δ and $\mathbf{u}$, we have that

$$\sum_{\mathbf{x} \oplus \mathbf{y} = \mathbf{z}} \langle A_x B_y \rangle = \sum_{\mathbf{x} \oplus \mathbf{y} = \mathbf{z}} (-1)^{\mathbf{u} \cdot (\mathbf{x} + \mathbf{y}) \oplus \delta}$$
$$= 2^n (-1)^{\mathbf{u} \cdot \mathbf{z} \oplus \delta} = 2^n \langle C_{z_1} \ldots C_{z_n} \rangle. \tag{24}$$

It immediately follows from the above identity that the inequalities (22) are valid for the $(n, 2, 2)$ full-correlation polytope.

Let us now suppose that the Bell inequality $I_{n22}(n, f, \tilde{p}) \leq 2^{-n} k(n, f, \tilde{p})$ is not facet-defining. Then we can write $I_{n22}(n, f, \tilde{p}) = I^1_{n22}(n, f, \tilde{p}) + I^2_{n22}(n, f, \tilde{p})$ and $k(n, f, \tilde{p}) = k^1(n, f, \tilde{p}) + k^2(n, f, \tilde{p})$ for some $I^1_{n22}(n, f, \tilde{p}), I^2_{n22}(n, f, \tilde{p}), k^1(n, f, \tilde{p}),$ and $k^2(n, f, \tilde{p})$ such that

$$I^1_{n22}(n, f, \tilde{p}) \leq 2^{-n} k^1(n, f, \tilde{p}) \tag{25}$$

and

$$I^2_{n22}(n, f, \tilde{p}) \leq 2^{-n} k^2(n, f, \tilde{p}) \tag{26}$$

are valid inequalities for the $(n, 2, 2)$ full-correlation polytope, i.e., they are satisfied by all deterministic points of the form (23). But then it follows from the above correspondence between deterministic point of the $(n, 2, 2)$ polytope and the $(2, 2^n, 2)$ polytope that $I(n, f, \tilde{p}) = I^1(n, f, \tilde{p}) + I^2(n, f, \tilde{p})$, where

$$I^1(n, f, \tilde{p}) \leq k^1(n, f, \tilde{p}) \tag{27}$$

and

$$I^2(n, f, \tilde{p}) \leq k^2(n, f, \tilde{p}) \qquad (28)$$

are valid NLC inequalities. This implies that $I(n, f, \tilde{p}) \leq k(n, f, \tilde{p})$ is not facet-defining for the $(2, 2^n, 2)$ polytope, from which the statement of the Lemma follows. $\square$

The above Lemma implies that it is sufficient to restrict our analysis on NLC inequalities associated with facet inequalities in the $(n, 2, 2)$-full correlation space. In Ref. [17] a construction for the coefficients $c(\mathbf{z})$ of all facet $(n, 2, 2)$ correlation Bell inequalities has been given. For small number of inputs, i.e. $n = 2, 3$, we have explicitly verified that none of the corresponding NLC inequalities are facet-defining; all these inequalities can actually be expressed as sums of CHSH inequalities. For larger n however, a similar analysis becomes difficult due to the large number of facet $(n, 2, 2)$ Bell inequalities and the high dimensionality of the $(2, 2^n, 2)$ correlation space.